# Performance models of correlators with random and systematic phase errors

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**Abstract.** Hybrid correlators are composed of numerous nonideal electronic and optical components that, to one degree or another, limit performance through unintended transformations of signals. Many of these effects show up as phase errors at a spatial light modulator (SLM) plane. The errors can be described as random variables, or as systematic offsets from the correct phases, as appropriate. Sources of systematic phase errors include quantizing circuits, incorrect or nonlinear amplifier gain, limited range phase modulators and residual phase modulation of amplitude-mostly SLMs. Random phase errors arise from electronic noise and fabrication variations of SLMs. Several systematic and random filter plane errors are related through a single parameter that describes the amount of phase mismatch. A model of peak-to-noise ratio (PNR) is also presented that describes the combined effects of random and systematic errors. This expression contains the products of two functions, one that depends only on systematic, the other on random, phase mismatch. PNR is also a function of the number of pixels in the filter plane modulator and a normalized moment of the amplitude of the image spectrum. The model is useful for developing phase error budgets for correlation systems.

*Subject terms: optical correlation; spatial light modulators; phase errors; correlation metrics; pattern recognition; diffractive optics; laser speckle.*

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## 1 Introduction

Optical correlation is highly dependent on the amount of phase mismatch or wavefront error across the filter plane. If the filter perfectly conjugates or matches the incident wavefront, then essentially all the energy is focused into a single bright spot at the correlation plane. If the filter poorly matches the incident wave, then the filter behaves like a diffuser in that the correlation intensity is greatly reduced and that a broad background of scattered light is produced akin to a speckle pattern.[1] It has been noted for various types of phase errors that small to moderate errors, say less than a quarter wave, only slightly reduce correlation intensity and only slightly increase the level of background noise in the correlation plane.[2,3] A quite similar situation is that the resolution of imaging systems is nearly diffraction limited if the total aberrations are less than one quarter wavelength.[4] Just as in imaging, the critical issue in optical correlation is the total amount of phase error rather than the functional description of the phase error. The physical reason for this is that the formation of a correlation peak is caused by the coherent superposition of a large number of wavefronts. Each cell or pixel in the filter plane produces at the correlation plane a wavefront with a specific amount of phase error and thus, the amount of filter plane phase error, rather than its spatial distribution controls the intensity of the correlation peak.

We have recently begun modeling the coherent formation of the correlation peak as a statistical process. The model is applicable even when the phase errors are nonrandom (i.e., systematic). Our earlier work on phase errors considered systematic[5] and random[6–8] phase errors individually. In this paper, we now are able to make direct comparisons between the relative effects of various systematic and random phase errors. The model can also be used to calculate the combined effects of systematic and random errors. Whereas the earlier models focused on phase-only filters, the models here apply to fully complex fractional power filters[9] (including the phase-only filter).

## 2 Correlation in the Presence of Filter Phase Errors

To focus on the effect of filter plane phase errors on correlation we model the idealized situation of a signal being correlated with a distorted version of itself. All distortions are produced by the spatial light modulator at the filter plane of the classical correlator and there are no other sources of noise or clutter. The optical correlator is translation invariant so it is sufficient to consider only the case where there is no coordinate shift between the signal and the impulse response of the filter. Thus, in this analysis the correlation plane response $c(x)$ peaks at $x=0$.

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The Fourier plane spectrum of the input plane signal $s(x)$ is written

$$S(f) = a'(f)\ \exp[j\phi(f)]\ , \qquad (1)$$

where $a'(f)$ is the positive valued amplitude spectrum and $\phi(f)$ is the phase spectrum. The filter transmittance is of the form

$$T'(f) = [a'(f)]^n\ \exp[-j\phi(f) + j\delta\phi(f)]\ , \qquad (2)$$

where $\delta\phi$ is the phase error introduced by the SLM and the exponent $n$ allows one to consider any fractional power filter,[9] including the phase-only filter for $n=0$. It is worth emphasizing, that although the amplitude and phase of many current SLMs are known to be coupled,[10] we are not considering this possibility in this analysis. Multiplying the two spectra [Eq. (1) and (2)] together produces the transmitted spectrum

$$T(f) = [a'(f)]^{n+1}\ \exp[j\delta\phi(f)] \equiv a\ \exp(j\delta\phi)\ , \qquad (3)$$

where $a = |T|$.

The results developed in the following can be directly applied to a 2-D array of $N$ pixels for which the pixels are regularly spaced with pitch $\Delta_{fx}, \Delta_{fy}$ in the frequency plane coordinates $f_x$ and $f_y$, and for which each pixel has clear area $w$. To simplify explanation, however, the SLM is modeled as a 1-D array of $N$ pixels, each of finite aperture of width $w$ and regularly spaced with pitch $\Delta_f$. (For examples of applying 1-D models to 2-D problem see Refs. 5 and 8.) If the illumination $S(f)$ contains no spatial frequencies that exceed half the sampling rate of the pixel array it is reasonable to assume that the transmitted spectrum can be described as a sampled function of the form

$$T(f) = \sum_{i=1}^{N} a_i\ \exp(j\delta\phi_i)\ \mathrm{rect}\left(\frac{f - \Delta - i\Delta_f}{w}\right)\ , \qquad (4)$$

where $a_i$ and $\delta\phi_i$ are the transmitted amplitude and phase at the $i$'th pixel. Value of the offset $\Delta$ is set to center the transmitted spectrum on the optical axis. The transmitted light diffracts into the complex amplitude distribution in the correlation plane according to the Fourier transform relationship

$$c(x) = \mathcal{F}[T(f)] \equiv \sum_i A_i(x)\ , \qquad (5)$$

where $A_i$ is the complex amplitude of the diffraction pattern from the $i$'th pixel. The amplitude of the resulting correlation peak is

$$c(0) = \sum_i a_i\ \exp(j\delta\phi_i) = Nw\overline{a\ \exp(j\delta\phi)}\ . \qquad (6)$$

Centering of the transmitted spectrum (via $\Delta$) in Eq. (4) eliminates a phase shift in Eq. (6) that is unimportant in our analysis. The second equality in Eq. (6) shows that the peak amplitude corresponds to the spatial average (indicated by overline) of $T(f)$ multiplied by $Nw$ the active transmitting area of the SLM. If there are no phase errors the correlation peak is a direct measure of $\bar{a}$ the average amplitude of $T(f)$. Equation 6 makes clear that the peak amplitude is independent of the spatial distribution of the phase errors across the filter plane, as initially discussed in Sec. 1.

## 2.1 *Influence of Systematic Phase Errors on the Correlation Peak*

One class of systematic phase errors is modulation-dependent errors.[5] These phase errors can be described as an explicit, nonrandom function of the form $\delta\phi(\phi;k)$, where phase error is a function of the value of the signal phase $\phi$ and a parameter $k$ that selects one function from a class of similar functions. For example, if the SLM only produces $-k\phi$ for each value $\phi$ from the signal spectrum then the phase error is

$$\delta\phi(\phi;k) = (1-k)\phi\ . \qquad (7)$$

We refer to this type of phase error as a gain or slope error. The parameter $k$ indicates to what degree the SLM phase matches the phase of the signal spectrum and $1\text{-}k$ indicates the magnitude of the phase errors. Various phase errors that result from systematic mappings of the desired phase $\phi$ to the actual phase produced by the SLM are defined in Table 1. The first column of the table describes the mapping relationship and the second column gives the expression for systematic phase errors (including nonunity slope error) as functions of $k$. Note for the case of phase quantized SLMs, which are commonly described in terms of the number of levels of quantization $m$, we have defined an equivalent phase match parameter $k$ using the relationship[5] $m = 1/(1-k)$.

A reasonable first-order approximation to Eq. (6), the correlation peak amplitude is[5]

$$c(0;k) \approx \frac{Nw\bar{a}}{2\pi} \int_{-\pi}^{\pi} \exp[j\delta\phi(\phi;k)]\ d\phi\ , \qquad (8)$$

which follows from modeling the amplitudes $a_i$ and the phases $\phi_i$ of the transmitted spectrum as random variables. Specifically, the approximation follows if the amplitudes and phases are statistically independent of each other, and if the phases $\phi_i$ are uniformly distributed over $2\pi$. The errors introduced by this approximation can be surprisingly small as compared to what one actually finds when digital simulating the optical correlation process using image of real-world objects, such as tanks and faces.[5] The closeness of the approximation to the exact result depends on the actual statistics or histogram of the occurrences of $(a_i, \phi_i)$. Equation (8), however, is often preferable to an exact result because it can be evaluated in closed form for many cases of interest. Closed-form expressions for Eq. (8), the effect of four specific types of systematic phase error on $c(0)$ are given in the third and fourth columns of Table 1. The third column is expressed in terms of commonly used parameters, e.g., $m$, if available. The fourth column is written in terms of $k$. Expressing correlation peak magnitude in terms of a single parameter is used to more directly compare the effect of each systematic (and also random) phase error in Sec. 2.3.

Note also that the magnitude of Eq. (8) is unchanged if there is a constant offset in the phase error. This has been used to simplify some of the expressions for phase error in Table 1. For the case of a phase quantized SLM, the phase error varies between 0 and $2\pi/m$. One might initially assume that quantizing so that the phase error varies between $-\pi/m$ and $\pi/m$ would produce a larger correlation peak. Because the constant $\exp(j\pi/m)$ can be factored outside the

**Table 1** Definitions of various systematic phase errors and relation of their effects through the parameter $k$.

| PHASE MAPPING | $\delta\phi(k)$ | $\lvert c(0)\rvert / Nw\bar{a}$ | $\lvert c(0;k)\rvert / Nw\bar{a}$ |
|---|---|---|---|
| Saturated at $\pm k\pi$ | $0 \quad ; 0 \le \lvert\phi\rvert \le k\pi$<br>$\phi - k\pi \,\mathrm{sgn}\,\phi \quad ; k\pi \le \lvert\phi\rvert \le \pi$ | | $k + (1-k)\,\mathrm{sinc}\,(1-k)$ |
| Non-unity slope of $k$ | $(1-k)\,\phi$ | | $\mathrm{sinc}\,(1-k)$ |
| Quantized in $m$ levels | $\mathrm{mod}\,[\phi, 2\pi(1-k)]$ | $\mathrm{sinc}\,\frac{1}{m}$ | $\mathrm{sinc}\,(1-k)$ |
| Binarized at 0 and $k\pi$ | $\phi \quad ; 0 \le \phi \le k\pi$<br>$(1-k)\,\phi \quad ; k\pi \le \phi \le 2\pi$ | $\mathrm{sinc}\,\frac{1}{2}\,\sin\frac{k\pi}{2}$ | $k\,\mathrm{sinc}\,\frac{k}{2}$ |

**Table 2** Definitions of various random phase errors and relation of their effects through the parameter $k$.

| RANDOM ERRORS | pdf $(\delta\phi)$ | $\lvert\langle c(0)\rangle\rvert / Nw\bar{a}$ | $\lvert\langle c(0;k)\rangle\rvert / Nw\bar{a}$ |
|---|---|---|---|
| Uniform of spread $\nu$ | $\frac{1}{\nu}\,\mathrm{rect}\,\frac{\delta\phi}{\nu}$ | $\mathrm{sinc}\,\frac{\nu}{2\pi}$ | $\mathrm{sinc}\,(1-k)$ |
| Gaussian of standard deviation $\sigma$ | $\frac{1}{\sqrt{2}\,\sigma}\,\exp\frac{-1}{2}\left(\frac{\delta\phi}{\sigma}\right)^2$ | $\exp\frac{-\sigma^2}{2}$ | $\exp\frac{-\pi^2}{6}\,(1-k)^2$ |

integral in Eq. (8), however, there is no change in the magnitude of the correlation peak. This property has been used to remove two additive terms of $\pi/m$ from the expression for phase error.

### 2.2 *General Analysis of the Influence of Random Phase Errors on the Correlation Peak*

Another class of phase errors is random. We specifically consider the case in which the phase errors are independent and identically distributed random variables, and the amplitudes are nonrandom. Furthermore, spatial variation of the transmitted spectrum over the aperture of a pixel is neglected [see Eq. (4)]. We derive expressions for $\langle c(0)\rangle$, the expected peak correlation amplitude; $\langle I_c(0)\rangle$, the expected intensity of the correlation peak; and $\sigma_I(0)$, its standard deviation, where $\langle\ \rangle$ represents the ensemble average or expectation operator.

The desired expressions follow from equations developed by Cohn and Liang that describe diffraction from an array of pixels in an illumination plane to a Fourier transform plane.[8] The equations were developed under the assumption that the light transmitted through any pixel is statistically independent of the light at any other pixel. No other assumptions were made about the spatial or statistical properties of the light transmitted through the SLM. The equations are more general than the present analysis in that (1) the SLM transmittance and the transmitted spectrum amplitude can vary with position across the aperture of a pixel, (2) both amplitude and phase can be random variables, and (3) the statistics from pixel to pixel can be nonidentical.

These general equations are rewritten here to specifically represent diffraction from the filter plane to the correlation plane. Equation (5) is the general form of the correlation plane amplitude. [The additional spatial description provided by Eq. (4) is not needed at this point. It will be used in Sec. 2.3.] Taking the expectation of Eq. (5) gives the expected correlation amplitude

$$\langle c(x)\rangle = \sum_i \langle A_i(x)\rangle \ . \tag{9}$$

The general expression for the expected correlation plane intensity is

$$\begin{aligned}\langle I_c(x)\rangle &= \sum_i \sum_j \langle A_i(x) A_j^*(x)\rangle \\ &= \lvert\langle c(x)\rangle\rvert^2 + \sum_i [\langle \lvert A_i\rvert^2\rangle - \lvert\langle A_i\rangle\rvert^2] \ .\end{aligned} \tag{10}$$

This expression separates into $\lvert\langle c(x)\rangle\rvert^2$ plus additional terms that result from the term $A_i(x)$ not being statistically independent from itself (i.e., terms in the double summation for which $i=j$). The general expression for the standard deviation of the correlation plane intensity distribution under the assumption that the $A_i(x)$ are statistically independent is

$$\begin{aligned}\sigma_I^2 &= \langle I_c\rangle^2 - 2\lvert\langle c\rangle\rvert^4 + \lvert\langle c\rangle^2 + \sum_i (\langle A_i^2\rangle - \langle A_i\rangle^2)\rvert^2 \\ &+ 4\mathrm{Re}[\langle c^*\rangle \sum_i (\langle \lvert A_i\rvert^2 A_i\rangle - \langle A_i^2\rangle\langle A_i^*\rangle + 2\lvert\langle A_i\rangle\rvert^2 \langle A_i\rangle - 2\langle\lvert A_i\rvert^2\rangle\langle A_i\rangle)] \\ &+ \sum_i [\langle\lvert A_i\rvert^4\rangle - 6\lvert\langle A_i\rangle\rvert^4 + 8\langle\lvert A_i\rvert^2\rangle\lvert\langle A_i\rangle\rvert^2 - \lvert\langle A_i^2\rangle\rvert^2 - 2\langle\lvert A_i\rvert^2\rangle^2 \\ &+ 4\ \mathrm{Re}(\langle A_i^2\rangle\langle A_i^*\rangle^2 - \langle\lvert A_i\rvert^2 A_i\rangle\langle A_i^*\rangle)] \ ,\end{aligned} \tag{11}$$

which was originally derived by evaluating

$$\langle I_c^2\rangle = \sum_i \sum_j \sum_k \sum_l \langle A_i A_j^* A_k^* A_l\rangle = \langle I_c\rangle^2 + \sigma_I^2(x) \ . \tag{12}$$

The specific results for the on-axis amplitude, intensity, and standard deviation when $T(f)$ is of the form of Eq. (4) are not given until Sec. 2.3, where they are directly found by setting the systematic errors to zero in subsequent equations that describe combined random and systematic errors.

## 2.3 *Influence of Random and Systematic Phase Errors on the Correlation Peak*

The deviation of phase of the transmitted wavefront at the filter plane is of the form $\delta\phi=\delta\phi_s+\delta\phi_r$, where the subscripts indicate the systematic and random phase errors. Using the preceding results and assumptions the expected peak correlation amplitude becomes

$$\langle c(0)\rangle = w\langle\exp(j\delta\phi_r)\rangle\sum_i a_i \exp(j\delta\phi_{is})$$

$$\approx N w\bar{a}\ \overline{\exp(j\delta\phi_s)}\ \langle\exp(j\delta\phi_r)\rangle \ . \tag{13}$$

The approximation in Eq. (13) follows from Eqs. (6) and (9). Equation (13) shows that the correlation amplitude is proportional to the product of the average systematic and the average random error phasors.

### 2.3.1 *Influence of random phase errors alone*

If the systematic phase errors are set to zero, then Eq. (13) describes the effect of specific random phase errors on correlation peak amplitude. Results for two specific types of random phase errors, uniform and Gaussian distributed, are given in Table 2. These distributions are commonly described in terms of the spread $v$ for the uniform and the standard deviation $\sigma$ for the Gaussian. The probability density function pdf($\delta\phi$) is given in the second column of Table 2 and $\langle c(0)\rangle$ is given in the third column in terms of $v$ and $\sigma$. Each parameter has been transformed into $k$, the degree of phase match (in units of wavelengths), and these results are presented in the fourth column. For the Gaussian case, $\sigma$, the standard deviation for the Gaussian distribution, has been defined in terms of $k$ by using the proportionality between spread $v$ and $\sigma_u$, standard deviation of the uniform distribution ($\sigma_u^2=v^2/12$).

### 2.3.2 *Comparing effects of various phase errors in terms of the phase match parameter k.*

Both systematic and random average error phasors can be evaluated in closed form for a number of specific cases of interest. Six cases are shown in Tables 1 and 2. Comparisons can easily be made, especially for the cases of nonunity slope, quantization, and uniform random phase errors, which have results of identical form. For the saturated case, the effect can be described as a linear combination of $k$ multiplied by $c(0;1)$, the correlation amplitude without phase errors, and $1$-$k$ multiplied by $c_n(0;k)$, the correlation amplitude for the nonunity slope characteristic. An additional correspondence between the quantized and binarized cases is brought out in the third column of Table 1. The result for quantized phase was calculated assuming that quantization is in steps of $2\pi/m$, whereas the result for binarized phase considers the phase levels to be other than 0 and $\pi$. The third column for the binarized case is written in the form of the correlation amplitude for quantized case when $m=2$ multiplied by a sine function. Written in this way, it can be seen that the reduction in correlation amplitude is slight for a separation between phase levels that is somewhat less than $\pi$. In all cases, as $k$ decreases from unity, correlation amplitude decreases from its maximum of $Nw\bar{a}$.

### 2.3.3 *Expected peak intensity and its standard deviation*

The expected intensity of the correlation peak follows from Eqs. (10) and (13) as

$$\langle I_c(0)\rangle = |\langle c(0)\rangle|^2 + Nw^2 q\overline{a^2}$$

$$=(Nw\bar{a})^2(pp_s+qZ/N) \ , \tag{14}$$

where the shorthand

$$p=\langle\exp(j\delta\phi_r)\rangle^2,$$

$$p_s=\overline{\exp(j\delta\phi_s)}^{\,2},$$

$$q=1-p,$$

$$Z=\frac{\overline{a^2}}{\bar{a}^2} \ , \tag{15}$$

is used. In Eq. (14), the only terms that depend on phase error are $p_s$, $p$, and $q$. Note that Tables 1 and 2 give the dependence of $p_s$ and $p$ on the phase match parameter $k$ because these terms are simply squares of the expressions in the fourth columns of the tables. For purposes of analyzing the effects of combined phase errors we will distinguish between the parameters describing systematic and random phase errors using the symbols $k$ and $k_r$, respectively.

Equation (14) is composed of two additive terms. The first term is the intensity of the desired coherent correlation peak in Eq. (13). The second term can be identified with the average intensity of incoherent noise or speckle background.[1] The relative magnitude of the coherent term will exceed that of the incoherent term as long as the phase errors and $Z$, a normalized moment of the transmitted amplitude spectrum, are not too large. The values of the parameter $Z$ range from a minimum value of unity if the amplitude spectrum is uniform across the SLM to a maximum value of $N$ if only one of the $N$ pixels is transmitting. A unity value of $Z$ corresponds to the case of inverse filtering; i.e., a fractional power filter with $n=-1$.

The standard deviation of the correlation peak is found by evaluating Eq. (11) along with the results in Eqs. (13) to (15) to get

$$\sigma_I^2(0)=(Nw\bar{a})^4\left\{2\frac{Z}{N}[q+(d-p)d_s]pp_s+\left[\frac{Z}{N}(d-p)d_s\right]^2 + 4\frac{Z_3}{N^2}(p-d-q)pp_s+\frac{Z_4}{N^3}(3p-6p^2+4pd-d^2-q)\right\} , \tag{16}$$

where we use the additional shorthand notation

$$d=\langle\exp(j2\delta\phi_r)\rangle,$$

$$d_s=\overline{\exp(j2\delta\phi_s)},$$

$$Z_l=\frac{\overline{a^l}}{\bar{a}^l} . \tag{17}$$

## 3 Model of Peak-to-Noise Ratio

To illustrate the combined effects of random and systematic phase errors more clearly we develop a model expression for the peak correlation amplitude to noise ratio[9,11] (PNR). We start with the general expression[5] of

$$\mathrm{PNR}^2=\left\{\frac{\langle I_c(0)\rangle}{\frac{1}{B_x-\Delta_x}\left[\int_{-B_x/2}^{B_x/2}\langle I_c(x)\rangle\,\mathrm{d}x-\int_{-\Delta_x/2}^{\Delta_x/2}\langle I_c(x)\rangle\,\mathrm{d}x\right]}\right\} , \tag{18}$$

where the denominator represents the root-mean-square amplitude across a correlation plane of spatial bandwidth $B_x$, which excludes a small region of width $\Delta_x$ centered around the correlation peak. With the filter plane SLM modeled as an array of equally spaced pixels of pitch $\Delta_f$, the nonredundant bandwidth is $B_x=\Delta_f^{-1}$; i.e., the spacing between diffraction orders of the array. The width $\Delta_x=B_x/N$ corresponds to the resolution of an $N$-element array and these definitions also lead to $B_f=N\Delta_f=\Delta_x^{-1}$. These choices have been made specifically so that our model closely corresponds to typical fast-Fourier-transform-based computer simulations that represent each filter plane pixel with one sample. Consistent with these assumptions, we approximate the second integral in the denominator of Eq. (18) as $\Delta_x\langle I_c(0)\rangle$. The first integral is the energy in the central diffraction order. It can be related to the energy in the filter plane through Parseval's theorem. The total energy in the filter plane and the correlation plane is

$$E_f=\int_{-B_f/2}^{B_f/2} a^2(f)\,\mathrm{d}f=B_f\overline{a^2}=E_c . \tag{19}$$

The energy in the central diffraction order is found by recognizing that the ratio of energy in each diffraction order is determined from the Fourier series of a square wave grating of duty cycle $D=w/\Delta_f$, where $w$ is the width of each pixel. Therefore the first integral in Eq. (18) can be written as $D^2E_f$. Using these results, Eq. (18) can be written as

$$\mathrm{PNR}=\left[\frac{(N-1)\langle I_c(0)\rangle}{N^2w^2\overline{a^2}-\langle I_c(0)\rangle}\right]^{1/2} . \tag{20}$$

Using Eq. (14) for $\langle I_c(0)\rangle$ in Eq. (20) completes the derivation of PNR (this expression is given presently). The PNR measured in an actual correlator will be subject to statistical fluctuations. One way to estimate the sensitivity of the measured PNR to randomness is to perturb $\langle I_c(0)\rangle$ the expected intensity in Eq. (20) by $\pm\sigma_I(0)$ from Eq. (16), and then use Eq. (14) to get

$$\mathrm{PNR}\pm\delta\mathrm{PNR}=\left\{\frac{(N-1)[pp_s+qZ/N\pm\sigma_I(0)/(Nw\bar{a})^2]}{Z-[pp_s+qZ/N\pm\sigma_I(0)/(Nw\bar{a})^2]}\right\}^{1/2} . \tag{21}$$

Setting the standard deviation to zero in Eq. (21) gives the expression for PNR. Equation (21) for the PNR and the perturbed PNR has no dependence on pixel width or duty cycle. [The factors of $Nw\bar{a}$ in Eq. (21) cancel with those in $\sigma_I(0)$; see Eq. (16).] Equation (21) also shows that PNR essentially depends on the square root of the number of pixels in the SLM. The effect on PNR of the term $Z$, which characterizes the spatial pattern of attenuation across the SLM, is nearly reciprocal to that of $N$. This relationship between $N$ and $Z$ is quite accurate if $N>>Z>>1$ and it can even be accurate for $Z=1$ if the phase errors are large [i.e., if the denominator term in brackets in Eq. (21) is much less than unity]. This relationship has led us to define $(N-1)/Z$ as an "effective number" of SLM pixels that are uniformly illuminated (this definition is discussed further in Refs. 3 and 8).

## 4 Example Analysis of Combined Systematic and Random Errors

The combined effect on PNR of nonunity slope (described by parameter $k$) and uniformly distributed random phase (described by the parameter $k_r$) is calculated using Eq. (21) [together with the definitions in Eqs. (15) to (17)]. The analysis is for a $128\times128$ SLM, i.e., $N=16{,}384$ and a transmitted amplitude spectrum of $Z=6.23$. These values were chosen to enable comparisons with the results given in Ref. 5 on systematic errors. The calculated results are presented in Figs. 1 through 5.

In Fig. 1, the curve for $k_r=1$ is identical to the modeled results for the case of nonunity slope in Ref. 5. For large values of $k$ the other curves are depressed by a factor of roughly $p^{1/2}$. This follows from the term $Z$ being much larger than $pp_s$ and $pp_s$ being much larger than $qZ/N$ for large values

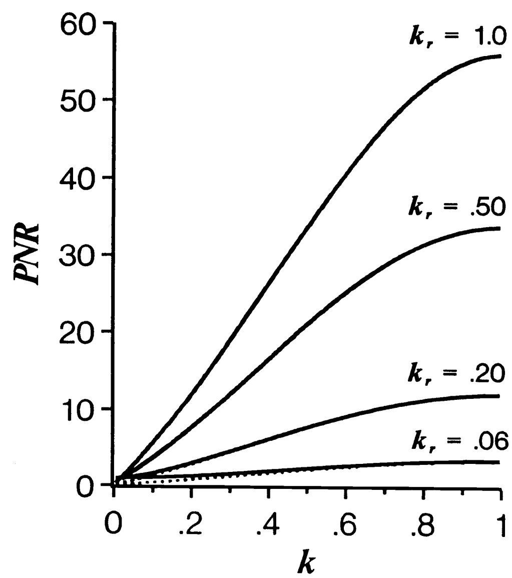


**Fig. 1** PNR as a function $k$ for various values of $k_r$; PNR is plotted as solid lines and its approximation as dotted lines.

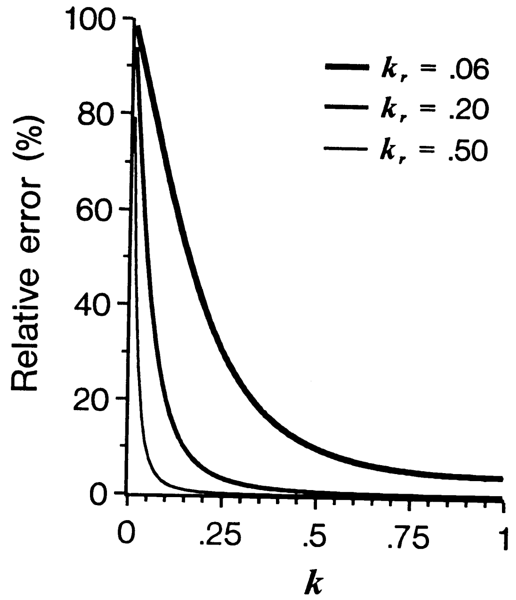


**Fig. 2** Relative error between PNR and the approximation to PNR plotted as a function of $k$ and for the same values of $k_r$ as in Fig. 1.

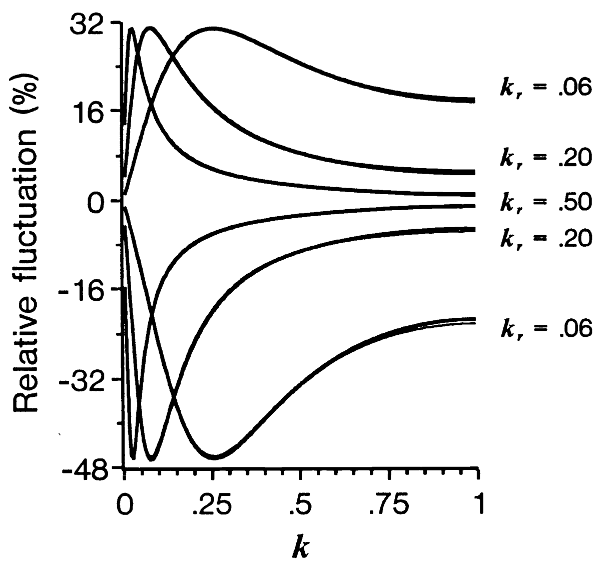


**Fig. 3** Fluctuation in PNR resulting from perturbation of intensity by plus and minus the standard deviation of the expected intensity plotted as a function of $k$ and for the same values of $k_r$ as in Figs. 1 and 2; fluctuation is shown as heavy lines and approximations to fluctuations are shown as light lines.

of $k$. The term $qZ/N$ can be practically ignored in this example, because the dashed curves that represent the values of PNR calculated with the term set to zero are only distinguishable at the lowest levels of PNR. In Fig. 2, the difference between PNR and its approximation is plotted as relative error with respect to PNR. The largest relative error is for $k=0$. At this point, PNR equals one and its approximation equals zero.

Figure 3 shows $\pm\delta\text{PNR}/\text{PNR}$ the relative fluctuation for the PNR curves in Fig. 1. The smallest fluctuations are found for values of $k$ and $k_r$ near unity. As the values of these parameters decrease, the fluctuation increases, but only to a point. Below that point the fluctuation decreases. The reason for the decrease is that the intensity of the correlation peak is being dominated by noise or speckle background. The noise-dominated region occurs when $pp_s$ [which is proportional to $|\langle c(0)\rangle|^2$] is less than $qZ/N$ (which is proportional to the average intensity of the noise background). The turning point in each of the curves occurs when $pp_s$ is of the same order of magnitude as $qZ/N$. The light lines in Fig. 3 are approximations in which all but the first of the four terms inside the braces in Eq. (16) have been omitted in the calculations. The differences are barely distinguishable and are primarily due to the second of the four terms. (We originally used values of $Z_3=Z^{1.5}$ and $Z_4=Z^2$, but numbers a few orders of magnitude larger cause no noticeable difference). What is interesting is that correlation performance can be predicted with such a small amount of specific information about the amplitude spectrum of the signal. It is also interesting that the percentage fluctuation in PNR is quite low for substantial amounts of random and systematic phase errors. This is a result of the coherent superposition of wavefronts from the typically large number of pixels in current SLMs.

An alternate way to plot Eq. (21) is to define $k_r$ in terms of $k$. In particular, we have chosen to plot PNR for a constant ratio of phase mismatch

$$\alpha=\frac{1-k_r}{1-k} . \tag{22}$$

That is, as the slope of the phase error $1-k$ increases, the random phase spread $v=2\pi(1-k_r)$ increases proportionally. This is plotted in Fig. 4 for various values of $\alpha$. For instance, for the case $\alpha=k_r/k=1$, PNR is reduced over the nonrandom case by no more than 40% for $k$ greater than 0.5. The corresponding fluctuation curve in Fig. 5 shows that random fluctuation resulting from one-standard-deviation perturbation in the expected intensity of the correlation peak is less than around 3% for $k$ greater than 0.5. To relate the magnitude

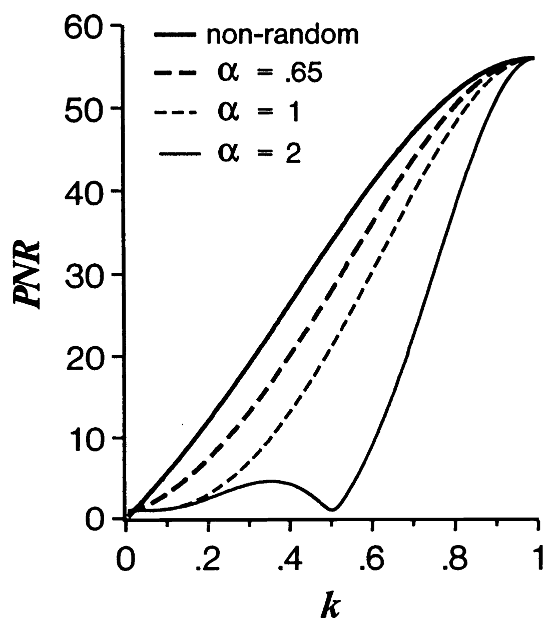


**Fig. 4** PNR shown as a function of $k$ for various values of $\alpha$, a constant proportional to the ratio of random to systematic phase mismatch.

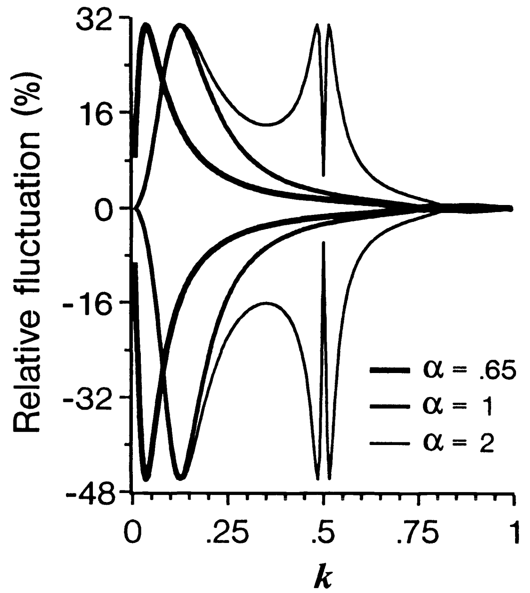


**Fig. 5** Relative fluctuation of PNR resulting from perturbation of the expected intensity by plus and minus the standard deviation of intensity plotted as a function of $k$ for the same values of $\alpha$ as in Fig. 4.

of these effects to well-known optical correlators note that for $k=0.5$ the model also describes the systematic phase errors produced by a binary phase-only filter (see Table 1). The PNR for the binary filter is reduced by roughly 40% over that of the analog phase-only filter. Phase randomness with a uniform spread over $\pi$ (i.e., $\alpha=1$) reduces the PNR for the binary case by another 40%. The random fluctuations are also useful in establishing the maximum level for the decision threshold. For example, for this noise-corrupted correlator, dropout or misclassification of the correlation peak is highly unlikely for thresholds set 10% below PNR [corresponding to a perturbation from the expected peak intensity of roughly $-3\sigma_I(0)$].

Also note in Fig. 5 that the fluctuation is essentially zero for the $\alpha=2$ curve at $k=0.5$. This is not a noise-immune operating point. Instead, PNR has dropped to a value of one and thus (as discussed for Fig. 3) the expected correlation peak intensity is small compared to the average intensity of the noise/speckle background.

## 5 Conclusions

We have developed a model that describes the performance of optical correlators subject to a combination of phase errors. Various types of phase errors are related through a single parameter $k$. Their effects on peak correlation amplitude are given in Tables 1 and 2. The effect on correlation amplitude of other systematic and random phase errors of interest can also be evaluated [using Eq. (8)], and in many cases, reduced to a functional form. This result can then be inserted in the model for PNR [Eq. (21) with the standard deviation set to zero] and the statistical fluctuation of PNR [Eq. (21) along with evaluation of the standard deviation using Eqs. (16) and (17)].

It was possible to develop such a simple method by decoupling any dependence between the phases and the amplitudes in the filter plane. When this is done, the influence of the transmitted amplitude spectrum is totally contained in the single parameter $Z$, which is the normalized second moment of the amplitude spectrum. This approximate model provides much insight and is useful at the early stages of design. It should be considered prior to performing exhaustive simulations and design studies on the effects of various phase errors. As part of this analysis, it may be desirable to consider the values of $Z$ for the objects in the image training set.

Of course, for many SLMs the values of phase and amplitude are coupled.[10] We did not consider this problem here. It seems reasonable, however, that there are alternate evaluations or approximations of Eq. (6) that can lead to models of nearly comparable usefulness and simplicity.

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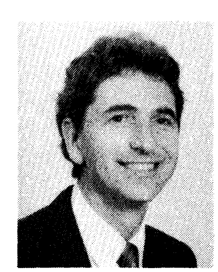

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